Weak Trefftz Discontinuous Methods for the resolution of frequency vibrations of composite structures

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ABSTRACT

This work is dedicated to the Weak Trefftz Discontinuous Methods, an extension of the Variational Theory of Complex Rays (VTCR), for the resolution of frequency vibrations of composite structures. This Weak Trefftz Discontinuous Methods allows one to build easily hybrid finite element / Trefftz strategies. Numerical illustrations are presented.

1 INTRODUCTION

The use of numerical simulation techniques has become an indispensable part of the industrial design process of innovative constructions for vibration performances. The Galerkin Finite Element Method (FEM) [1] is a well-established tool, which is commonly used for the analysis of vibration problems. However, as it uses continuous, piecewise polynomial shape functions, it often leads to huge numerical model. As a consequence, in practice, its use is restricted to low-frequency range applications. Trefftz methods [2] have been proposed as a means to bypass this limitation. They differ from the FEM in the expansion of the field variables, as they use shape functions that are exact solutions of the governing differential equations. Compared to the FEM, Trefftz methods often lead to a considerable reduction in model size and computational effort. Some examples of such methods are: a special version of the partition of unity method [3], the ultra weak variational method [4], the plane wave discontinuous Galerkin method [5], the least-squares method [6], the discontinuous enrichment method [7], the element-free Galerkin method [8], the wave boundary element method [9] and the wave-based method [10]. The Variational Theory of Complex Rays (VTCR), first introduced in [11] for steady-state vibration problems and in [12] for 3-D acoustics problems, also belongs to that category. The main differences between these methods lie essentially in the treatment of the transmission conditions at the boundaries of the elements or substructures.

The VTCR uses a specific weak formulation of the problem, which enables the approximations within the substructures (the shape functions that verify the governing equations) to be a priori independent of one another. Thus, any type of shape function can be used within a given substructure provided it satisfies the governing equation, thus giving the approach great flexibility. In this work, we introduce extensions of the classical VTCR formulation: the constraint, which imposes the verification of the governing equation, is weakened. This leads to a new numerical method, which can be called the weak Trefftz discontinuous Galerkin method. These extensions allow one to easily couple different types of numerical models and then get hybrid models: the FEM and the classic VTCR models used together, for example. As a consequence, they lead to new approaches to the resolution of engineering problems where composite structures have to be faced.

2 DEVELOPMENT OF HYBRID METHODS

Our hybrid method will be presented on a Helmholtz vibration problem. Then, consider a standard problem defined on a domain Ω with boundaries $\partial\Omega = \partial_1\Omega \cup \partial_2\Omega$ (see Figure 1): find $u \in H^1(\Omega)$ such that

$$\begin{cases} (1+i\eta)\Delta u + k^2 u + r_d = 0 \quad \text{over} \quad \Omega \\ u = u_d \quad \text{over} \quad \partial_1\Omega \\ (1+i\eta)\partial_n u + hiku = g_d \quad over \quad \partial_2\Omega \end{cases}$$

where $\partial_n u = grad\ u.n$, (n being the outward normal). k is the wave number; h is a constant related to the vibration impedance; r_d and g_d are prescribed sources. The damping coefficient η is positive. The data are supposed to be sufficiently regular to have a unique solution.

Let us introduced the VTCR variational formulation for this problem. To do this, the domain Ω is divided into subdomains Ω_E. The interface between two subdomains E and E' is denoted $\Gamma_{EE'}$. The VTCR is a Trefftz approach which uses the affine space $U = \left\{ u / u \in U_E \text{ on } \Omega_E \right\}$ with

$$U_E = \left\{ u_E / u_E \in V_E \subset H^1(\Omega_E); (1+i\eta)\Delta u_E + k^2 u_E + r_d = 0 \text{ on } \Omega_E \right\}$$

The vector spaces (with $r_d = 0$) associated with U and U_E are denoted U_0 and $U_{E,0}$. We also denote $\left\{ u \right\}_{EE'} = (u_E + u_{E'})_{\Gamma_{EE'}}$ and $\left[u \right]_{EE'} = (u_E - u_{E'})_{\Gamma_{EE'}}$. Denoting $q_u = (1+i\eta) grad\, u$, the VTCR formulation can be written: find $u \in U$ such that

$$\begin{aligned} \mathrm{Re}\Bigg(-ik \Bigg(& \sum_{E,E'} \int_{\Gamma_{EE'}} \left(\frac{1}{2} \left\{ q_u.n \right\}_{EE'} \left\{ \bar{v} \right\}_{EE'} - \frac{1}{2} \left[\overline{q_v}.n \right]_{EE'} \left[u \right]_{EE'} \right) dS \\ & - \sum_E \int_{\Gamma_{EE} \cap \partial_1 \Omega} \overline{q_v}.n \left(u - u_d \right) dS \\ & + \sum_E \int_{\Gamma_{EE} \cap \partial_2 \Omega} \frac{1}{2} (-\overline{q_v}.n(u + (q_u.n - g_d)/(hik)) + \bar{v}(q_u.n + hiku - g_d)) dS \Bigg) \Bigg) \\ & = 0 \quad \forall v \in U_0 \end{aligned}$$

where the over line represents the complex conjugate part, and "Re" the real part. It is proven that this variational formulation is equivalent to the reference problem. All that has to be done to get an approximation is to replace U_E by the finite dimension subspace U_E^h in the variational formulation, for example spanned by N_E propagative waves $e^{ik(\theta).x}$ regularly reparted on the θ – polar $\left[0; 2\pi \right[$ range for 2-D examples.

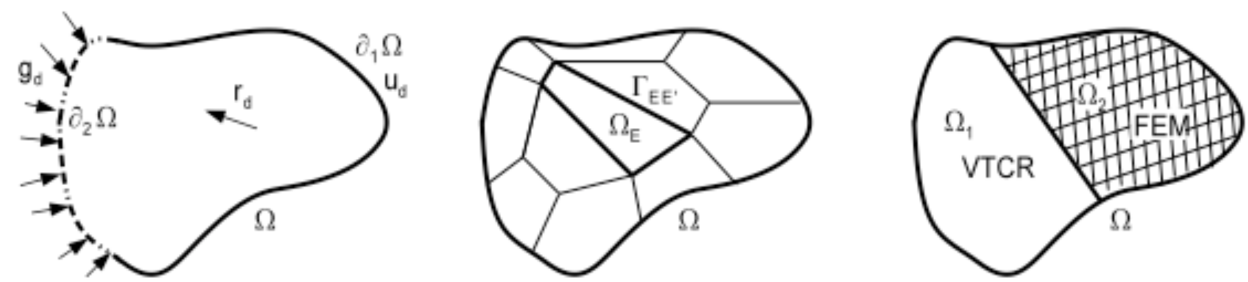

Figure 1: Left: definition of the computational domain. Middle: definition of the subdomains. Right: coupling of the FEM and VTCR descriptions for hybrid models.

However, one constraint with the VTCR is the need for verifying the governing equation (also called the Trefftz constraint): $u \in U$. As a consequence, FEM shape functions can not be used directly. Then, in order to develop hybrid methods, which mix VTCR and FEM approximations, one has to weaken this constraint (then leading to the definition of weak Trefftz methods). To do this, it is necessary to modify the variational formulation. The new variational formulation is now: find $u \in U$ (with now no constraint on U) such that:

$$\mathrm{Re}\left(-ik\left(\sum_{E,E'}\int_{\Gamma_{EE'}}\left(\frac{1}{2}\{q_u.n\}_{EE'}\{\overline{v}\}_{EE'}-\frac{1}{2}[\overline{q_v}.n]_{EE'}[u]_{EE'}\right)dS\right.\right.$$
$$-\sum_E\int_{\Gamma_{EE}\cap\partial_1\Omega}\overline{q_v}.n(u-u_d)\,dS$$
$$+\sum_E\int_{\Gamma_{EE}\cap\partial_2\Omega}\frac{1}{2}(-\overline{q_v}.n(u+(q_u.n-g_d)/(hik))+\overline{v}(q_u.n+hiku-g_d))\,dS$$
$$\left.\left.+\sum_E\int_{\Omega_E}(div\,q_u+k^2u+r_d)\overline{v}\,d\Omega\right)\right)=0\quad\forall v\in U_0$$

As one can see, the difference between this formulation and the last one is just the add of a new term which weaken the governing equation. Again, it can be demonstrated that this formulation is equivalent to the reference problem. As no constraint is needed for the definition of the space U, FEM approximation can be used anywhere it is needed. We can also mix the approximations by using the VTCR approximation in a partition Ω_1 and the FEM approximation in another partition Ω_2 (see Figure 1). As a consequence, a great flexibility is provided by such an approach, especially on composite structures, where VTCR or FEM can be used (see [13] for the VTCR).

3 NUMERICAL ILLUSTRATION

We here consider the example described in Figure 2 for the illustration of hybrid methods. The problem is considered to be homogeneous, and restricted to a 2-D L-curve shape domain. The wave numbers of the fluids a and b are $k_a = 6.5\,\mathrm{m}^{-1}$ and $k_b = 29.4\,\mathrm{m}^{-1}$. The damping coefficients are $\eta_a = \eta_b = 0.001$. The boundary conditions are Robin condition with $h = 0.001$ and $g_{\boldsymbol{d}} = 0$ or $1\,\mathrm{m}^{-1}$.

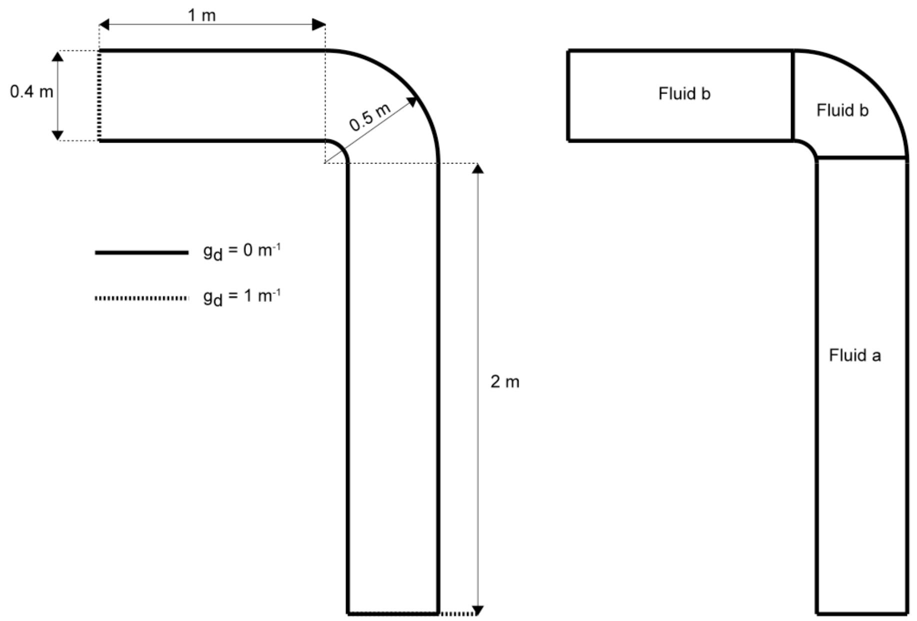

Figure 2: Definition of the computational domain for the illustration example used in Section 3.

The selected subdomaining for computing the problem can be seen in Figure 3. The FEM approximation has been used in Ω_1, where the wavelength is the smallest. Its mesh uses 10 elements along the x-axis and 40 elements along the y-axis, leading to a mesh with 451 DOFs. The VTCR approximation has been used in the subdomains Ω_E with $E \in \{2..13\}$. In each of these subdomains, the number N_E of the regularly oriented used rays satisfies $N_E = \left[2.k.diam(\Omega_E)\right]$ where $[\square]$ stands for the integer part, k stands for the wavenumber and $diam(\Omega_E)$ stands for the diameter of Ω_E. With such a choice, we are sure that the VTCR subdomains contains enough DOFs to get a good solution (see the heuristic criterion in [12]). The corresponding result can be seen on Figure 3. One can see that, globally, the solution looks like the FEM reference solution visible in Figure 3, and computed with a very refined mesh. Indeed, almost all the vibration peaks are located at the right location and have the right amplitude. Then, this example validates the proposed approach and shows that the hybrid method can solve this very complex numerical example, which mixes different kind of approximation with different kind of physics.

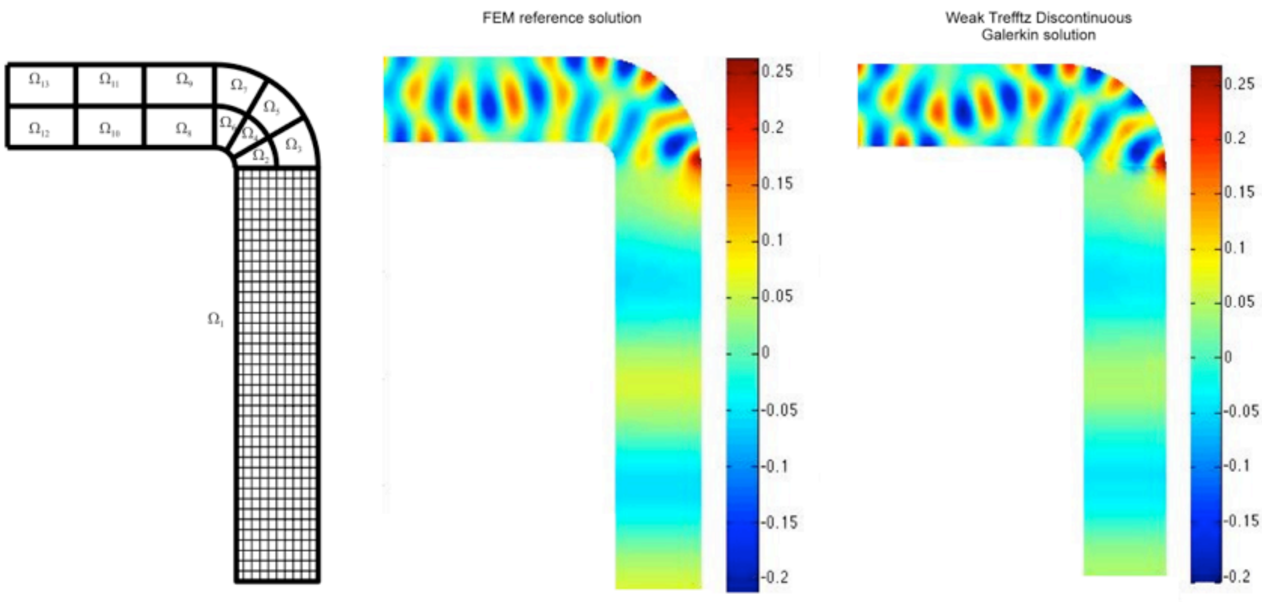

Figure 3 Left: selected subdomaining and modeling for computing the problem of Section 3, defined in Figure 2. Middle: FEM reference solution. Right: numerical result obtained with the proposed hybrid approach.

4 CONCLUSION

In the VTCR proposed in [11], [12] and [13], the solution of a vibrational problem is tough in an approximated space spanned by exact solutions of the governing equation (i.e. propagative waves). Their continuities along the interface between the subdomains are ensured thanks to a dedicated variational formulation. Here, this variational formulation is extended in such a way that any kind of approximation can be used: exact solutions of the governing equation, or not. Then, hybrid approximation mixing VTCR and FEM approximation can be used. This has been illustrated on an example, which mixes two types of fluid. Waves are used to approximate the vibrational response of the fluid which contains the smallest wavelength. Polynomial FEM shape functions are used to approximate the response the fluid which contains the largest wavelength. As a consequence, this work is dedicated to the definition of a new generation of computational strategy, able to easily mix different kind of approximations, which is useful for the prediction of the vibrational performances of composite structures.

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