CONSTITUTIVE LAW MISFIT FUNCTIONAL FOR CAVITIES IDENTIFICATION

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ABSTRACT

The aim of this work is an analysis of a geometric inverse problem related to the identification of defaults, namely cavities in structures. The approach proposed in treating such problem is to transform it into a shape optimization one by the means of an error functional that can be interpreted as a constitutive law misfit functional. Moreover, this cost functional whose shape derivative is explicitly determined allows the implementation of a numerical minimization algorithm using the gradient information combined with the level set method. The efficiency of the proposed approach is highlighted by numerical results.

1 INTRODUCTION

Throughout this paper, we consider a bounded domain $B \subset \mathbb{R}^2$ with boundary Υ, occupied by a linear elastic material and we assume that there exists a cavity, namely a bounded domain $\overline{A} \subset B$ with boundary Γ. Let us denote by Ω the domain $B \setminus \overline{A}$. The forward linear elastic problem is therefore given by

$$\left\{\begin{array}{lcll} \operatorname{div}\sigma(u) &=& 0 & \text{in } \Omega, \\ \sigma(u) &=& \lambda\, \mathrm{tr}\,\varepsilon(u)\; I + 2\,\mu\,\varepsilon(u) & \text{in } \Omega, \\ \sigma(u)\; n &=& 0 & \text{on } \Gamma, \\ \sigma(u)\; n_\Upsilon &=& g & \text{on } \Upsilon, \end{array}\right. \tag{1}$$

where u is the displacement, $\sigma(u)$ is the associated stress tensor, $\varepsilon(u)$ is the linearized strain tensor given by $\varepsilon(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$. n_Υ and n are the outward unit normals to the boundary of Ω. The geometric inverse problem under consideration consists so in recovering the cavity A, namely the unknown shape Γ by applying some prescribed load g on Υ and measuring the induced displacement on the same part Υ, i.e

$$\left\{\begin{array}{lcll} u &=& f & \text{on } \Upsilon, \\ \sigma(u)\; n_\Upsilon &=& g & \text{on } \Upsilon. \end{array}\right.$$

For a given Ω, let u_D and u_N be the solutions of the following Dirichlet, respectively Neumann problem

$$\left\{\begin{array}{lcll} \operatorname{div}\sigma(u_D) &=& 0 & \text{in } \Omega, \\ \sigma(u_D) &=& \lambda\, \mathrm{tr}\,\varepsilon(u_D)\; I + 2\,\mu\,\varepsilon(u_D) & \text{in } \Omega, \\ \sigma(u_D)\, n &=& 0 & \text{on } \Gamma, \\ u_D &=& f & \text{on } \Upsilon, \end{array}\right. \tag{2}$$

respectively

$$\left\{\begin{array}{lcll} \operatorname{div}\sigma(u_N) &=& 0 & \text{in } \Omega, \\ \sigma(u_N) &=& \lambda\, \mathrm{tr}\,\varepsilon(u_N)\; I + 2\,\mu\,\varepsilon(u_N) & \text{in } \Omega, \\ \sigma(u_N)\, n &=& 0 & \text{on } \Gamma, \\ \sigma(u_N)\, n_\Upsilon &=& g & \text{on } \Upsilon. \end{array}\right. \tag{3}$$

Thus, the cavities identification problem can be formulated as a shape optimization one (see [1, 2, 4]) as follows

$$\left\{\begin{array}{l} \text{Find } \Omega \text{ such that} \\ J(\Omega) = \min\limits_{\tilde{\Omega} \subset B} J(\tilde{\Omega}), \end{array}\right. \tag{4}$$

using the constitutive law misfit functional

$$J(\Omega) := \frac{1}{2}\int_\Omega (\sigma(u_D) - \sigma(u_N)) : (\varepsilon(u_D) - \varepsilon(u_N)). \tag{5}$$

The main contribution of the present work relies on the use of the error functional (5) that can be interpreted as an energetic least-squares one.

2 SHAPE DERIVATIVE

We consider a hold-all domain $U \supset \overline{\Omega}$ and construct a family of perturbations F_t as follows

$$F_t = id + th,$$

where h is a deformation field belonging to the space

$$Q = \{h \in \mathcal{C}^{1,1}(\overline{U})^2;\ h = 0 \text{ on } \Upsilon\}$$

and t is sufficiently small such that F_t is a diffeomorphism from Ω onto its image. The family of domains $\{\Omega_t\}$ respectively $\{\Gamma_t\}$ are then defined by $\Omega_t := F_t(\Omega)$ respectively $\Gamma_t := F_t(\Gamma)$. The condition $h_{|\Upsilon} = 0$ means that the boundary Υ is a part of the boundary of Ω_t.

Definition 1 *The Eulerian derivative of the functional J at Ω in the direction of an element $h \in Q$ is defined by the quantity, when it exists*

$$J'(\Omega, h) = \lim_{t \mapsto 0} \frac{J(\Omega_t) - J(\Omega)}{t}.$$

The Eulerian derivative is called shape derivative if $J'(\Omega, h)$ exists for all $h \in Q$ and the mapping $h \mapsto J'(\Omega, h)$ is linear and continuous with respect to the topology of $\mathcal{C}^{1,1}(\overline{\Omega})^2$.

Theorem 1 *The mapping $t \mapsto J(\Omega_t)$ is $\mathcal{C}^1$ in a neighborhood of 0 and its derivative at 0 is given by*

$$J'(\Omega, h) = \int_\Gamma G\,(h \cdot n)\,,$$

with

$$G = \frac{1}{2}\left[(\sigma(u_D) : \varepsilon(u_D)) - (\sigma(u_N) : \varepsilon(u_N))\right]. \tag{6}$$

3 NUMERICAL RESULTS

An iterative method is proposed to solve the shape optimization problem (4). The last theorem allows to choose like a descent direction of the functional J

$$h \in Q \qquad \text{such that} \qquad h_{|\Gamma} = -G\,n,$$

where G is given by (6). To numerically implement this iterative process, we use the level set method [3].

3.1 First case

As a first test, we consider $\Upsilon = \{x;\ |x| = 0.9\}$. The solution is the circle centered at the origin with radius equal to 0.35. The convergence is obtained after 17 iterations as it is shown in Figure 1.

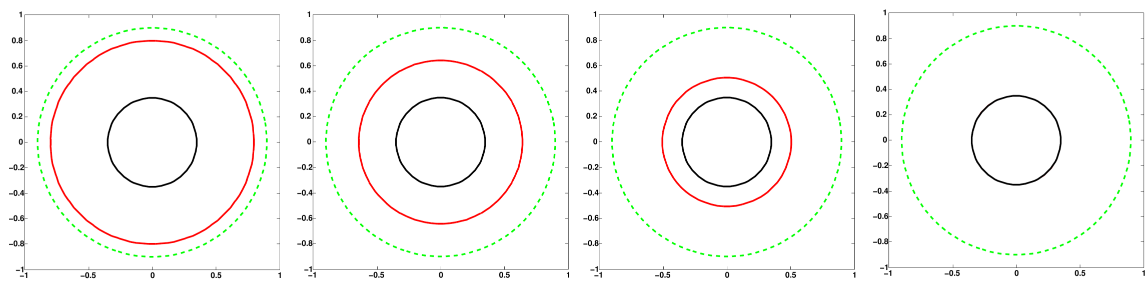

Figure 1: Υ the exterior boundary (the dashed green line), Γ the exact solution (the black line), evolution of the boundary Γ^k (the red line) for $k = 0, 10, 14, 17$ (left to right).

3.2 Second case

In this second case, the cavity to recover is a connected domain, namely the disc of radius equal to 0.3 centered at the origin. However, we consider a disconnected initial guess. Indeed, Γ^0 is the union of the three disjointed circles of radius R equal respectively to 0.14, 0.17 and 0.14 (left to right), centered respectively at $(-0.4, 0)$, $(0, 0)$ and $(0.4, 0)$ as it is shown in Figure 2. The convergence is obtained after 9 iterations.

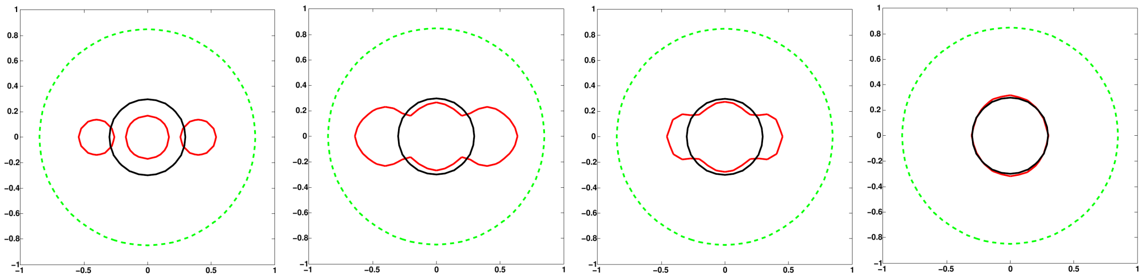

Figure 2: Topology change test: Υ the exterior boundary (the dashed green line), Γ the exact solution (the black line), evolution of the boundary Γ^k (the red line) for $k = 0, 1, 3, 9$ (left to right).

4 CONCLUSION

In this work, a cavities identification problem in linear elasticity was transformed to a shape optimization one by the means of a Dirichlet-Neumann misfit functional. To solve this problem, we made use theoretically of the shape derivative concept and numerically of the level set method. The numerical tests illustrated the efficiency of the proposed approach.

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