A WAVE BASED UNIT CELL METHOD TO PREDICT ABSORPTION AND TRANSMISSION COEFFICIENTS OF POROELASTIC MATERIALS CONTAINING PERIODIC INCLUSIONS

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ABSTRACT

This paper presents an efficient Wave Based modelling procedure to predict the absorption and transmission coefficient of infinite poroelastic materials containing a periodic grid of inclusions. As compared to standard numerical prediction schemes it offers the following advantages: (1) contrarily to Transfer Matrix Methods the layers do not need to be homogeneous, (2) contrarily to multipole methods, the inclusions do not need to be circular, (3) contrarily to element based prediction techniques, unbounded domains can easily be accounted for. Moreover, the procedure allows more easily for optimisation routines since it is a meshless and computationally more efficient technique. The Wave Based Method is an indirect Trefftz approach; it approximates the dynamic fields using a weighted sum of exact solutions of the governing differential equations. The Multi-Level Wave Based Method, which allows to describe the dynamic field of a cavity containing an inclusion, is extended in two ways: (1) Bloch-Floquet conditions are imposed on the boundaries to take into account the periodicity of the complete structure and (2) novel unbounded acoustic wave functions are presented that fulfil the acoustic Helmholtz equation, the Sommerfeld radiation condition and the Bloch-Floquet conditions. The implementation of the method is validated with the multipole method.

1 INTRODUCTION

In many industrial applications poroelastic materials are applied as efficient noise reduction measures. These materials show to be most effective in the mid to high frequency range where the acoustic wavelengths are of the same order of magnitude as the thickness of the material. In the past decades, much research effort is spent in order to increase the absorption of the material in the low frequency range. Often multilayered structures are applied in order to prohibit wave propagation, combining different kinds of porous and viscous materials combined with air gaps. Although effective, this approach may lead to heavier and more bulky solutions. Another approach consists in studying inhomogeneous materials, such as double porosity materials [1] or poroelastic materials containing inclusions [2]. This paper focuses on the numerical modelling of the latter.

An efficient Wave Based modelling procedure is presented to predict the absorption and transmission coefficient of laterally infinite poroelastic materials containing a periodic grid of inclusions. The Wave Based Method (WBM) [3, 4] is an indirect Trefftz approach; it approximates the dynamic fields using a weighted sum of exact solutions of the governing differential equation(s). It can be applied to any dynamic problem of which mathematical description of the governing physics can be cast into a (number of) Helmholtz equation(s). The unknowns are the contribution factors of the wave functions. A sufficient condition for the WBM to converge is that the considered problem domains are convex. Non-convex domains need to be partitioned into a (preferably small) number of subdomains. When considering geometries containing inclusions, it is clear that the standard WBM cannot easily be applied as it would lead to many subdomains. For circular inclusions it would even be impossible. To overcome these constraints, the Multi-Level WBM (ML-WBM) has been developed [5]. The bounded domain and each of the inclusions are considered in a different 'level' as if the others are not present. Their approximation sets are then combined using a weighted residual approach.

When considering an acoustic plane wave impinging on an infinite 2D poroelastic material with periodic inclusions, the response of the entire structure is characterized by the response of a unit cell. The theorem of Bloch states that the relative amplitude change and phase shift of a wave propagating through an infinite periodic structure, is the same across each cell; as a result the response of any unit cell can be expressed in terms of the response of a reference unit cell multiplied by an exponential term that defines the amplitude and the phase shift as the wave propagates from the reference cell to neigbouring cells.

In this paper, the ML-WBM is extended in two ways: (1) Bloch-Floquet conditions are imposed on the boundaries to take into account the periodicity of the complete structure and (2) novel unbounded acoustic wave functions are presented that fulfil the acoustic Helmholtz equation as well as the Sommerfeld radiation condition and Bloch-Floquet conditions. As compared to standard numerical prediction schemes it offers the following advantages: (1) contrarily to Transfer Matrix Methods the layers do not need to be homogeneous, (2) contrarily to multipole methods, the inclusions do not need to be circular, (3) contrarily to element based prediction techniques, unbounded domains can easily be accounted for. Moreover, the procedure is meshless such that it more easily allows for optimization routines.

The implementation of the method is validated with the multipole method. Two examples show the effect of different types of inclusions on the absorption as well as the transmission coefficient.

2 PROBLEM DESCRIPTION

The mathematical problem setting of a general 2D periodic coupled (semi-) infinite acoustic-poroelastic steady-state problem containing rigid circular inclusions, as shown in Figure 1 is given in this section. A time-harmonic motion with $e^{j\omega t}$-dependence is assumed.

The problem domain Ω can be divided into two non-overlapping domains Ω_a and Ω_e, containing air and a poroelastic medium, described as an equivalent fluid, respectively. A plane wave is impinging on the poroelastic structure, incident at an angle θ. The thickness of the poroelastic structure is denoted L_y and the heterogeneities are periodic in the x-direction with period L_x. In the application cases of this paper, the inclusions are considered rigid (i.e. Neumann boundary conditions) and circular. In a completely similar way, other boundary conditions can be introduced. Moreover, the Multi-Level WBM allows to study different inclusion geometries as well; there is no restriction to circular geometries.

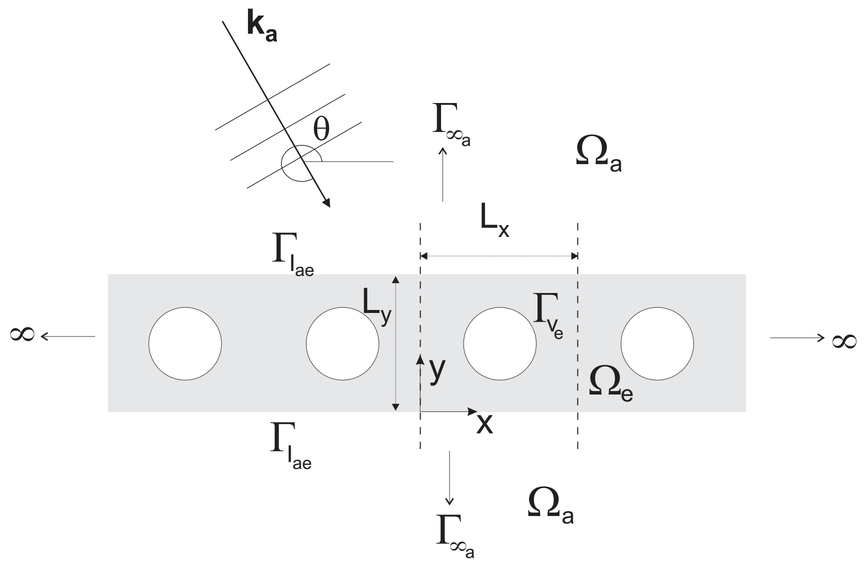

Figure 1: Problem description of a 2D infinite poroelastic structure with periodic circular inclusions, coupled to two infinite acoustic domains.

It is assumed that the poroelastic material has a rigid frame and can be modelled as an equivalent fluid medium, with a complex effective density ρ_e and effective compressibility K_e, following the same expressions as in [2] The steady-state pressure $p_e(\mathbf{r})$, inside the medium Ω_e, is governed by the Helmholtz equation. Also the acoustic pressure, $p_a(\mathbf{r})$, in medium Ω_a is governed by the Helmholtz equation. Consequently, the steady state pressure $p_\bullet(\mathbf{r})$, with $\bullet = a$ for the acoustic and $\bullet = e$ for the equivalent fluid case is given by:

$$\mathbf{r} \in \Omega_\bullet : \quad \nabla^2 p_\bullet(\mathbf{r}) + k_\bullet^2 p_\bullet(\mathbf{r}) = \mathcal{F}_\bullet(\mathbf{r}), \tag{1}$$

where, ∇^2 is the Laplacian operator, $k_\bullet = \omega / c_\bullet$ is the wave number of medium $\Omega_\bullet$. The fluid is excited by a source defined by $\mathcal{F}_\bullet(\mathbf{r})$.

Due to the geometrical periodicity and the plane wave nature of the excitation, the resulting dynamic fields have to be periodic in the x-direction as well. The dynamic fields in all cells can be related to the one of a single cell using the Bloch-Floquet relation [6]:

$$\forall\ N \in \mathbb{Z} : \quad p_\bullet(x + NL_x, y) = p_\bullet(x, y) e^{-jk_{ax} N L_x}, \tag{2}$$

where $k_{ax} = k_a \cos\theta$.

In this paper, only Sommerfeld, Neumann boundary conditions and coupling conditions between an equivalent fluid and an acoustic domains are considered. On the exterior acoustic boundary at infinity, Γ_{∞_a}, the former the condition applies:

$$\mathbf{r} \in \Gamma_{\infty_a} : \quad R_{\infty_a}(\mathbf{r}) = \lim_{|\mathbf{r}| \to \infty} \left(\sqrt{|\mathbf{r}|} \left(\frac{\partial p_a(\mathbf{r})}{\partial |\mathbf{r}|} + jk_a p_a(\mathbf{r}) \right) \right) = 0\ . \tag{3}$$

On the rigid boundaries determined by the inclusions, Γ_{v_e} the following residual applies:

$$\mathbf{r} \in \Gamma_{v_e}: \quad R_{v_e}(\mathbf{r}) = \mathcal{L}_{v_e}(p_e(\mathbf{r})) - \bar{v}_{e,n} = 0, \tag{4}$$

with $\bar{v}_{e,n}$ the prescribed value for the normal velocity, being 0 m/s for a rigid boundary. The velocity operator is defined as:

$$\mathcal{L}_{v_\bullet}(*) = \frac{\mathrm{j}}{\rho_\bullet \omega} \frac{\partial *}{\partial \gamma_{\mathbf{n}}}, \tag{5}$$

with $\gamma_{\mathbf{n}}$ the normal direction to the boundary, pointing outwards.

On the interfaces $\Gamma_{I_{ae}}$ between the acoustic and the poroelastic medium, the continuity of pressure and velocity are imposed:

$$\mathbf{r} \in \Gamma_{I_{ae}}: \begin{cases} R_{Ip_{ae}}(\mathbf{r}) = p_a(\mathbf{r}) - p_e(\mathbf{r}) = 0, \\ R_{Iv_{ae}}(\mathbf{r}) = \mathcal{L}_{v_a}(p_a(\mathbf{r})) + \mathcal{L}_{v_e}(p_e(\mathbf{r})) = 0. \end{cases} \tag{6}$$

The governing Helmholtz equations in the different domains (1), the periodicity conditions (2), the applied boundary conditions (3)-(4) and interface conditions (6) define a unique pressure field.

3 MULTI-LEVEL WAVE BASED METHOD

The main idea of the WBM Multi-Level approach [5] is to consider the different inclusions and the bounded domain as different 'levels' of the problem. Each level considers the scattering of one specific object, or the dynamic wave field within the bounded domain as if the other inclusions and/or the bounded domain were not present. The total solution field can then be obtained by combining the different levels together in a weighted residual approach, using the superposition principle. The concept is explained for the simple acoustic problem in Figure 2 showing a bounded problem domain with one circular inclusion. For a complete discussion, the reader is referred to [5].

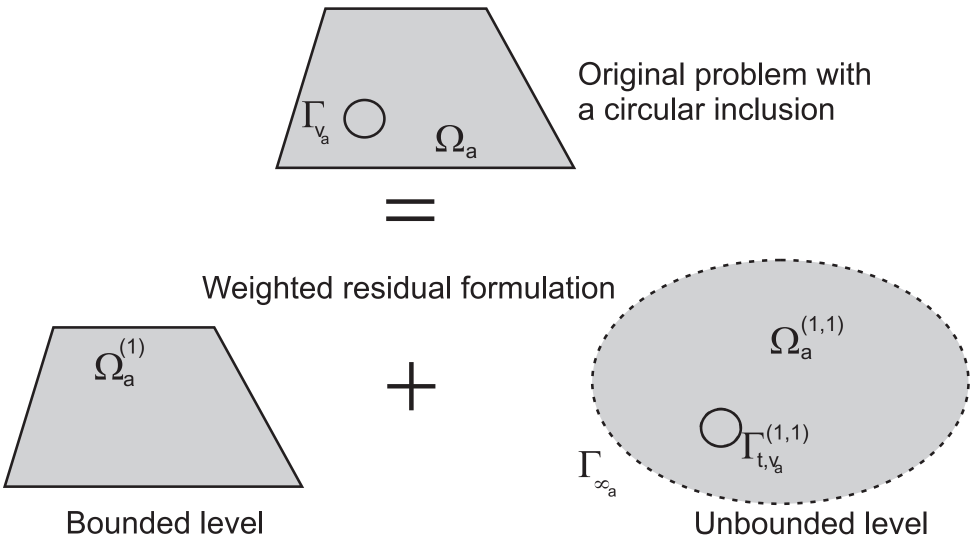

Figure 2: Graphical representation of the Multi-Level modelling concept for bounded problems.

The Multi-Level WBM approach consists of four steps that are briefly revisited for this simple problem setting:

1. *Division of the original problem into levels*
In a first step the original problem is divided into a number of levels: the first level includes the bounded problem as if there were no inclusions present. If the bounded domain is non-convex, it is further partitioned into convex subdomains (not needed in this case). This bounded subdomain is indicated $\Omega_a^{(1)}$ where subscript (1) indicates the index of the bounded subdomain. The other level considers the scattering due the inclusion as if the bounded domain was not present. The truncation circle $\Gamma_{t,v_a}^{(1,1)}$ circumscribes the inclusion (it this case it is coinciding with the actual boundary). The first subscript digit indicates the index of the bounded subdomain to which the inclusion belongs, the second indicates the index of the unbounded level. In this specific case only Neumann boundary conditions are applied on the truncation surface. The unbounded acoustic subdomain exterior to $\Gamma_{t,v_a}^{(1,1)}$ is denoted $\Omega_a^{(1,1)}$.

2. *Selection of wave functions for the different levels:*
For each subdomain belonging to a level, a suitable wave function set is selected to describe its dynamic field. Following the WBM procedure [4, 5], the acoustic pressure is approximated by a solution expansion $\hat{p}_a^{(\bullet)}(\mathbf{r})$:

$$p_a^{(\bullet)}(\mathbf{r}) \simeq \hat{p}_a^{(\bullet)}(\mathbf{r}) \ = \ \sum_{w=1}^{n_{a,w}^{(\bullet)}} p_{a,w}^{(\bullet)} \Phi_{a,w}^{(\bullet)}(\mathbf{r}) + \hat{p}_{a,q}(\mathbf{r}) = \mathbf{\Phi}_{\mathbf{a}}^{(\bullet)}(\mathbf{r}) \ \mathbf{p}_{\mathbf{a,w}}^{(\bullet)} + \hat{p}_{a,q}(\mathbf{r}). \tag{7}$$

The wave function contributions $p_{a,w}^{(\bullet)}$ are the weighting factors for each of the selected wave functions $\Phi_{a,w}^{(\bullet)}$. All weighting factors together form the vector of degrees of freedom $\mathbf{p}_{\mathbf{a,w}}^{(1)}$. The corresponding *a priori* defined wave functions are collected in the row vector $\mathbf{\Phi}_{\mathbf{a}}^{(1)}$. The particular solution $\hat{p}_{a,q}(\mathbf{r})$ accounts for the effect of source terms, resulting from an inhomogeneous Helmholtz equation (1). For this particular example, this term vanishes. For this example, $\bullet$ can be replaced by 1 for the bounded domain or 1,1 for the unbounded domain.

For the 2D bounded subdomain, two types of wave functions are distinguished, the so-called r- and s-set:

$$\sum_{w=1}^{n_{a,w}^{(1)}} p_{a,w}^{(1)} \Phi_{a,w}^{(1)}(\mathbf{r}) \ = \sum_{w_r=1}^{n_{a,w_r}^{(1)}} p_{a,w_r}^{(1)} \Phi_{a,w_r}^{(1)}(\mathbf{r}) + \sum_{w_s=1}^{n_{a,w_s}^{(1)}} p_{a,w_s}^{(1)} \Phi_{a,w_s}^{(1)}(\mathbf{r}) \tag{8}$$

with $n_{a,w}^{(\alpha)} = n_{a,w_r}^{(1)} + n_{a,w_s}^{(1)}$. These wave functions are defined as:

$$\Phi_{a,w}^{(1)}(x,y) = \begin{cases} \Phi_{a,w_r}^{(1)}(x,y) = \cos(k_{a,xw_r}^{(1)} x)\, e^{-\mathrm{j}k_{a,yw_r}^{(1)} y} \\ \Phi_{a,w_s}^{(1)}(x,y) = e^{-\mathrm{j}k_{a,xw_s}^{(1)} x}\, \cos(k_{a,yw_s}^{(1)} y) \end{cases}. \tag{9}$$

Desmet [3] has shown that the following selection of wave number components leads to a converging wave function set:

$$\begin{aligned} \left(k_{a,xw_r}^{(1)}, k_{a,yw_r}^{(1)}\right) &= \left(\frac{w_{1_a}^{(1)}\pi}{L_{x_a}^{(1)}}, \pm\sqrt{k_a^2 - \left(k_{a,xw_r}^{(1)}\right)^2}\right) \\ \left(k_{a,xw_s}^{(1)}, k_{a,yw_s}^{(1)}\right) &= \left(\pm\sqrt{k_a^2 - \left(k_{a,yw_s}^{(1)}\right)^2}, \frac{w_{2_a}^{(1)}\pi}{L_{y_a}^{(1)}}\right) \end{aligned} \tag{10}$$

with $w_{1_a}^{(1)}$ and $w_{2_a}^{(1)} = 0, 1, 2, \ldots$. The dimensions $L_{x_a}^{(1)}$ and $L_{y_a}^{(1)}$ represent the dimensions of the (smallest) bounding rectangle, circumscribing the considered subdomain $\Omega_a^{(1)}$.

The wave functions for the unbounded domain $\Omega_a^{(1,1)}$ are chosen to explicitly comply with not only the Helmholtz equation, but also with the Sommerfeld radiation condition at Γ_{∞_a}. The following wave function set for unbounded domains exterior to a circular truncation curve with radius R_{t_a} is used, distinguishing between a c- and an s-set:

$$\Phi_{a,w}^{(1,1)}(r,\theta) = \begin{cases} \Phi_{a,w_c}^{(1,1)}(r,\theta) = H^{(2)}_{w_{1_a}^{(1,1)}}(k_a r)\cos(w_{1_a}^{(1,1)}\theta) \\ \Phi_{a,w_s}^{(1,1)}(r,\theta) = H^{(2)}_{w_{2_a}^{(1,1)}}(k_a r)\sin(w_{2_a}^{(1,1)}\theta) \end{cases} \tag{11}$$

with $w_{1_a}^{(1,1)} = 0, 1, 2, \ldots$ and $w_{2_a}^{(1,1)} = 1, 2, 3, \ldots$ and $H_n^{(2)}(\bullet)$ is the n-th order Hankel function of the second kind. As for bounded domains, the series of functions (11) needs to truncated in order to be used in a numerical scheme. A similar truncation rule as for the bounded domains is used and determines the highest orders $w_{1_a,max}$ and $w_{2_a,max}$ of the Hankel functions used in the exterior wave function expansion [7]. We denote $\hat{p}_a^{(1,1)}(\mathbf{r})$ the field variable of the unbounded acoustic subdomain $\Omega_a^{(1,1)}$.

The pressure field $\hat{p}_a^{(1')}(\mathbf{r})$ in the compound subdomain $\Omega_a^{(1')}$=$\Omega_a^{(1)} \bigcap \Omega_a^{(1,1)}$ can be written as:

$$\mathbf{r} \in \Omega_a^{(1')}: \quad \hat{p}_a^{(1')}(\mathbf{r}) = \hat{p}_a^{(1)}(\mathbf{r}) + \hat{p}_a^{(1,1)}(\mathbf{r}). \tag{12}$$

3. *Construction of the system of equations:*
   The residuals on boundaries and interfaces are minimised using a weighted Galerkin approach and using the compound wave function set for subdomains $\Omega_a^{(1')}$. Different test functions are selected for the different boundaries. The only prerequisite to have valid test functions is that they need to be able to represent an arbitrary field on that specific boundary. As unbounded wave functions can accurately represent any field on the truncation surface they are associated to, they are used as weighting functions on that truncation boundary. A similar reasoning is followed for the bounded wave functions defined for $\Omega_a^{(1)}$: they can be used as weighting functions on the boundaries defining the bounded level.

4. *Solution and post-processing:*
   The system matrices can be solved for the unknown contribution factors of all wave functions. In a post-processing step, the response field can be evaluated.

4 WBM FOR PERIODIC STRUCTURES

As pointed out in Section 2 it is sufficient to study the response in one unit cell to be able to reconstruct the response in any point of the system. The bounded, poroelastic part of the problem domain can be modelled using the Multi-Level WBM using the equivalent fluid properties to determine the wave number components in the pressure expansions. The next subsections discuss the application of the periodicity conditions and the wave functions in the semi-unbounded periodic acoustic domains.

4.1 Bloch-Floquet boundary conditions in the multilevel WBM framework

The wave functions used in the Multi-Level WBM framework do not fulfil the periodicity condition, equation (2). The advantage of using the WBM is, however, that there is no restriction to circular inclusions as compared to the multipole method. Geometrically not too complex inclusions can be dealt with using the regular multilevel WBM.

The Bloch-Floquet periodicity conditions have to be embedded in the WBM in a weak sense. The following residuals are minimised on the left (Γ_{BFL_e}) and right boundary (Γ_{BFR_e}) of the poroelastic unit cell:

$$\mathbf{r} \in \Gamma_{BFL_e}: \quad R_{BFL_e}(\mathbf{r}) = p(\mathbf{r}) - p(\mathbf{r}^{\mathbf{r}}) e^{\mathrm{j}k_{ax}L_x} = 0, \tag{13}$$

$$\mathbf{r} \in \Gamma_{BFR_e}: \quad R_{BFR_e}(\mathbf{r}) = \mathcal{L}_{v_e}\left(p(\mathbf{r})\right) + \mathcal{L}_{v_e}\left(p(\mathbf{r}^{\mathbf{l}})\right) e^{-\mathrm{j}k_{ax}L_x} = 0, \tag{14}$$

in which $\mathbf{r}^{\mathbf{l}} \in \Gamma_{BFL_e}$ and $\mathbf{r}^{\mathbf{r}} \in \Gamma_{BFR_e}$.

4.2 Semi-unbounded Bloch-Floquet acoustic wave functions

A novel wave function set is needed in the semi-unbounded acoustic domains to avoid integration on infinite boundaries. Again, the pressure field can be approximated by a weighted set of wave functions, equation (7). Wave functions are selected that fulfil the Helmholtz equation, the Sommerfeld radiation condition and the Bloch-Floquet periodicity condition. The wave functions $\Phi_{a,w}^{(\alpha)}(\mathbf{r})$ for a semi-unbounded periodic domain are a plane wave expansion:

$$\Phi_{a,w}^{(\alpha)}(\mathbf{r}\,(x,y)) = e^{-\mathrm{j}\left(k_{BF_{x,w}}x + k_{BF_{y,w}}y\right)}. \tag{15}$$

The wave number components $k_{BF_{x,w}}$ are selected such that the periodicity condition is fulfilled:

$$k_{BF_{x,w}}^{(\alpha)} = k_{ax} + \frac{2m\pi}{L_x}, \tag{16}$$

with $m \in \mathbb{Z}$. In order to fulfil the Helmholtz equation, the wave numbers $k_{BF_{y,w}}$ are selected as:

$$k_{BF_{y,w}}(\alpha) = \pm\sqrt{k_a^2 - \left(k_{BF_{x,w}}^{(\alpha)}\right)^2}, \tag{17}$$

and the sign of the root is selected such that the waves are purely outgoing and consequently the Sommerfeld condition is fulfilled.

A plane wave source is exciting the system. For this source, the particular term yields:

$$\hat{p}_{a,q}(\mathbf{r}) = Ae^{-\mathrm{j}\mathbf{k_a}\cdot\mathbf{r}}, \tag{18}$$

with A the plane wave amplitude, $\mathbf{k_a} = (k_{ax}, k_{ay}) = (k_a \cos\theta, k_a \sin\theta)$ the wave vector and θ the propagation angle, see also Figure 1.

4.3 Reflection, transmission and absorption coefficient evaluation

Due to the plane wave nature of the wave functions in the semi-unbounded periodic acoustic domains, the hemispherical reflection and transmission coefficients, $\mathcal{R}$ and $\mathcal{T}$, can be calculated as:

$$\mathcal{R} = \sum_w \frac{\Re(k_{BF_{yw}}^{(1)})||p_{a,w}^{(1)}||^2}{k_{ay}||A||^2}, \tag{19}$$

$$\mathcal{T} = \sum_w \frac{\Re(k_{BF_{yw}}^{(2)})||p_{a,w}^{(2)}||^2}{k_{ay}||A||^2}. \tag{20}$$

The absorption coefficient $\mathcal{A}$ can be evaluated via:

$$\mathcal{A} = 1 - \mathcal{R} - \mathcal{T}. \tag{21}$$

5 NUMERICAL VALIDATION

The validation case considers the problem setting shown in Figure 3. The dimensions, the frequency range as well as the material properties are based on values available from literature [8].

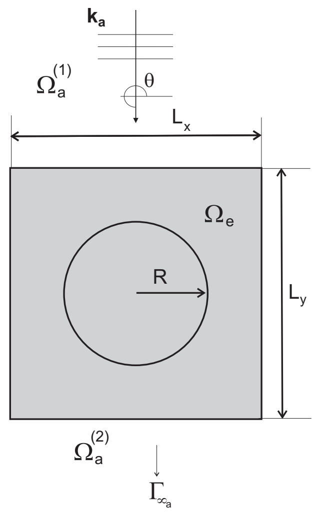

Figure 3: Acoustic-Poroelastic-Acoustic unit cell problem geometry with a circular rigid inclusion.

The angle of incidence is $3\pi/2$. The radius of the rigid inclusion R is taken to be 2.5 mm. The thickness L_y of the foam and the distance L_x in between the inclusions are both 1 cm.

Figure 4 shows the absorption, reflection, and transmission coefficient of the periodic medium, with and without periodic rigid circular inclusions, calculated with the WBM and the multipole method for frequencies between 1 kHz and 300 kHz. A perfect match between the WBM and the MPM is seen, confirming the validity of the former.

It is seen that the reflection coefficient is clearly increased due to the addition of the inclusions. Some of the incoming energy is reflected on the rigid inclusion instead of travelling through the material. The absorption coefficient is increased around the modified plate modes, leading to an entrapment of energy. Except around these frequencies, the absorption coefficient is not altered to a large extent. The increased reflection and absorption coefficient obviously have a beneficial effect on the obtained transmission coefficient.

6 CONCLUDING REMARKS

This paper discusses the extension of the Multi-Level Wave Based Method to predict the absorption, reflection and transmission coefficients of poroelastic structures with periodic inclusions. In this paper, as a first step, only circular inclusions are considered, although this is no limitation for the method. The poroelastic material is modelled as an equivalent fluid. Bloch-Floquet periodicity conditions are imposed in a weak sense, minimising residuals on the boundary conditions. A novel wave function set is applied for the acoustic semi-infinite domains; each wave function fulfills the acoustic Helmholtz equation, the Sommerfeld radiation condition and the periodicity conditions. The method has been applied for a simple sound transmission problem and the results obtained are confirmed by a multipole implementation. In a next step, the method will be applied in optimisation studies as the procedure is meshless and there is no restriction towards circular inclusions.

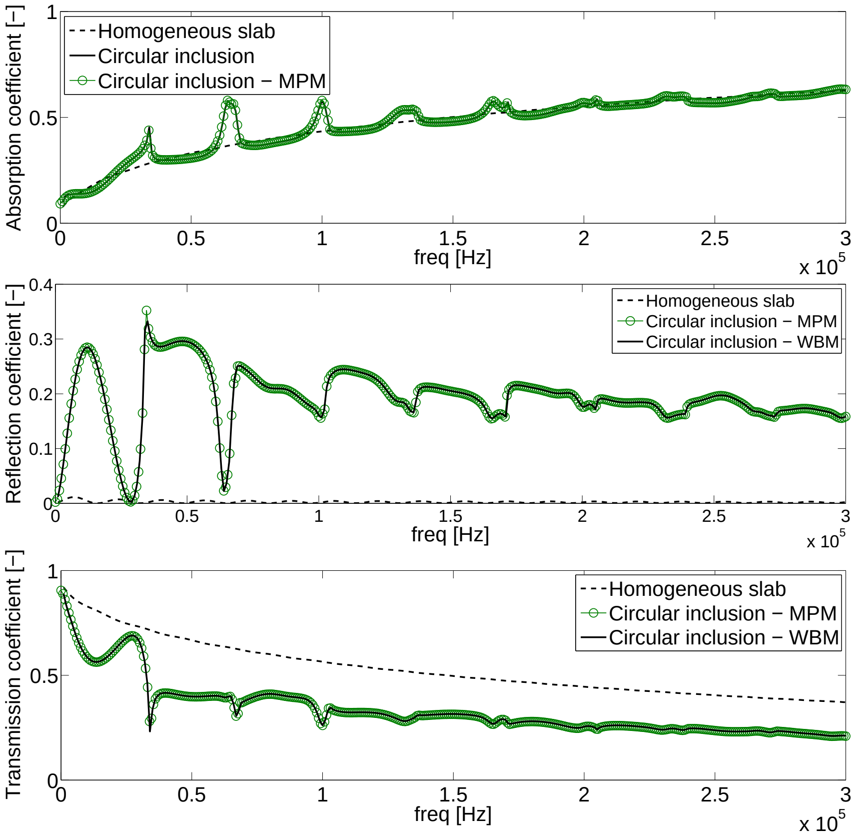

Figure 4: Absorption coefficient (top), reflection coefficient (middle) and transmission coefficient (bottom) for the polyurethane foam with and without circular inclusions.

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